

# Preddoktorski izpit (Preddoctoral exam)

25. aprila 2022 (25th April 2022)

Čas reševanja je 150 minut. Število možnih točk je 24. Za pozitivno oceno morate doseči vsaj 12 točk. Če ni izrecno povedano drugače, morate vse odgovore utemeljiti. Veliko uspeha!

The exam duration is 150 minutes. The total number of available marks is 24. To pass the exam, you must achieve a total of at least 12 marks. All answers must be justified, unless explicitly instructed otherwise. Good luck!

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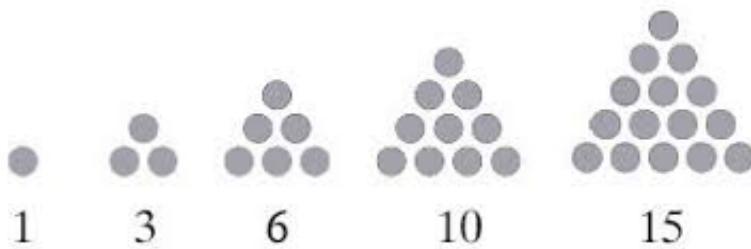
Ime in priimek

1	<input type="text"/>
2	<input type="text"/>
3	<input type="text"/> Sedež (3.06)
4	<input type="text"/>
5	<input type="text"/>
6	<input type="text"/> Vpisna številka
7	<input type="text"/>
8	<input type="text"/>
9	<input type="text"/>
10	<input type="text"/>
11	<input type="text"/>
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Rešite **vse štiri** od 1. do 4. naloge.

Answer **all four** of questions 1–4

## 1. naloga (3 točke/marks)



Zgornja slika prikazuje prvih 5 trikotniških števil. Določite vsa trikotniška števila, ki so praštevila.

The picture above illustrates the first 5 triangular numbers. Find all triangular numbers that are prime numbers.

**2. naloga (3 točke/marks)**

Naj bo  $f : \mathbb{R} \rightarrow \mathbb{R}$  dana s predpisom

$$f(x) = 4x \cos(2x) - 2 \sin(2x) + \pi.$$

- a) Dokažite, da ima  $f$  natanko eno ničlo na  $(0, \pi/2)$ .
- b) Dokažite, da ima  $f$  neskončno ničel na  $\mathbb{R}$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = 4x \cos(2x) - 2 \sin(2x) + \pi.$$

- a) Prove that  $f$  has exactly one zero in  $(0, \pi/2)$ .
- b) Prove that  $f$  has infinitely many zeros in  $\mathbb{R}$ .

**3. naloga (3 točke/marks)**

Naj bo  $\mathbb{F}_5$  končno polje reda 5. Koliko je avtomorfizmov vektorskega prostora  $\mathbb{F}_5 \times \mathbb{F}_5$ ?

Let  $\mathbb{F}_5$  be the finite field of order 5. How many automorphisms does the vector space  $\mathbb{F}_5 \times \mathbb{F}_5$  have?

**4. naloga (3 točke/marks)**

Naj bo  $f: \mathbb{R} \rightarrow \mathbb{R}$  taka konveksna funkcija, da je  $f(0) \leq 0$ . Dokažite, da za  $t, x \in \mathbb{R}$ , kjer je  $t \geq 1$ , velja

$$tf(x) \leq f(tx).$$

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a convex function such that  $f(0) \leq 0$ . Prove that, for any  $t, x \in \mathbb{R}$  with  $t \geq 1$ , we have

$$tf(x) \leq f(tx).$$

Rešite **natanko** dve od 5. do 11. naloge.

Answer **exactly two** of questions 5–11

### 5. naloga (6 točk/marks)

Naj bo  $\mathbb{F}$  končno polje s  $q$  elementi. Naj bo

$$\begin{aligned}\mathbb{F}[X]_1 &= \{p(X) \in \mathbb{F}[X] \mid \text{vodilni koeficient } p(X) \text{ je enak 1}\} \\ \text{Irr}_{\mathbb{F}}[X] &= \{p(X) \in \mathbb{F}[X]_1 \mid p(X) \text{ je nerazcepna}\}.\end{aligned}$$

Za  $s \in \mathbb{C}$  definirajmo

$$\zeta(s) = \sum_{p(X) \in \mathbb{F}[X]_1} \left( q^{\deg p(X)} \right)^{-s}.$$

Pokažite naslednje trditve:

a) Če je realni del števila  $s$  strogo večji od 1, potem je

$$\zeta(s) = \frac{1}{1 - q^{1-s}}.$$

b) Velja

$$\zeta(s) = \prod_{p(X) \in \text{Irr}_{\mathbb{F}}[X]} \left( 1 - \frac{1}{q^{s \deg p(X)}} \right)^{-1}.$$

c) Naj bo  $s \in (1, \infty)$ . Dokaži, da velja

$$\log \zeta(s) \leq 3\zeta(2s) + \sum_{p(X) \in \text{Irr}_{\mathbb{F}}[X]} q^{-s \deg p(X)}.$$

d) Sklepaj, da je  $\text{Irr}_{\mathbb{F}}[X]$  neskončna množica.

Let  $\mathbb{F}$  be a finite field with  $q$  elements. Define

$$\begin{aligned}\mathbb{F}[X]_1 &= \{p(X) \in \mathbb{F}[X] \mid \text{the leading coefficient of } p(X) \text{ is 1}\} \\ \text{Irr}_{\mathbb{F}}[X] &= \{p(X) \in \mathbb{F}[X]_1 \mid p(X) \text{ is irreducible}\}.\end{aligned}$$

For  $s \in \mathbb{C}$ , we define

$$\zeta(s) = \sum_{p(X) \in \mathbb{F}[X]_1} \left( q^{\deg p(X)} \right)^{-s}.$$

Prove the following statements.

a) If the real part of  $s$  is strictly greater than 1 then

$$\zeta(s) = \frac{1}{1 - q^{1-s}}.$$

b) It holds that

$$\zeta(s) = \prod_{p(X) \in \text{Irr}_{\mathbb{F}}[X]} \left( 1 - \frac{1}{q^{s \deg p(X)}} \right)^{-1}.$$

c) Suppose  $s \in (1, \infty)$ . Prove that

$$\log \zeta(s) \leq 3\zeta(2s) + \sum_{p(X) \in \text{Irr}_{\mathbb{F}}[X]} q^{-s \deg p(X)}.$$

d) Conclude that  $\text{Irr}_{\mathbb{F}}[X]$  is an infinite set.

## 6. naloga (6 točk/marks)

Dokažite, da obstaja natanko ena funkcija  $y \in C^1(\mathbb{R})$ , ki zadošča  $y(1) = 1$  in reši diferencialno enačbo

$$yy'e^{xy'} = xe^y.$$

Nasvet: poskusite uganiti preprosto rešitev.

Prove that there exists exactly one function  $y \in C^1(\mathbb{R})$  that satisfies  $y(1) = 1$  and solves the differential equation

$$yy'e^{xy'} = xe^y.$$

Hint: try to find a simple solution.

## 7. naloga (6 točk/marks)

Naj bo  $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$  padajoče zaporedje zaprtih podprostorov v kompaktnem Hausdorffovem prostoru  $X$  in naj bo  $F := \bigcap_i F_i$ . Dokažite naslednji trditvi:

- Za vsako odprto množico  $U$ , ki vsebuje  $F$ , obstaja tak  $i$ , da  $U$  vsebuje  $F_i$ .
- Če so vsi podprostori  $F_i$  povezani, potem je povezan tudi njihov presek  $F$ .

Let  $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$  be a sequence of nested closed subspaces of a compact Hausdorff space  $X$ , and let  $F := \bigcap_i F_i$ . Prove the following statements:

- For every open set  $U$  that contains  $F$ , there exists  $i$  such that  $U$  contains  $F_i$ .
- If all subspaces  $F_i$  are connected, then so is  $F$ .

## 8. naloga (6 točk/marks)

- Dokažite, da je vsak hamiltonov 3-regularni graf 3-povezavno-obarvljiv.
- Najdite primer 3-regularnega grafa, ki je 3-povezavno-obarvljiv ampak ni hamiltonov.
- Prove that every Hamiltonian 3-regular graph is 3-edge-colourable.
- Find an example of a 3-regular graph that is 3-edge-colourable but not Hamiltonian.

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f(A,s,t,q) : let r = random[s,t] // a uniformly random integer r with s <= r <= t
               let p = A[r]
               let k = partition(A,s,t,p)
               if s + q - 1 < k then return f(s,k-1,q)
               else if s + q - 1 > k then return f(k+1,t,q-k+s-1)
               else return A[k]

```

Slika 1:

### 9. naloga (6 točk/marks)

V spodnjih točkah je  $A$  tabela celih števil, ki je definirana za vse indekse  $i$  v intervalu  $s \leq i \leq t$ . Velikost tabele označimo z  $n$ , tj.  $n = t - s + 1$ .

- a) Napišete algoritem  $\text{partition}(A,s,t,p)$ , kjer je  $p = A[i]$  za nek indeks  $i$  v intervalu  $s \leq i \leq t$ . Algoritem mora permutirati vrednosti tabele  $A$  *na mestu* (tj., končna permutacija mora biti zapisana v isti tabeli  $A$ ) in vrniti tako celo število  $k$  z intervala  $s \leq k \leq t$ , da veljajo spodnje trditve.
- i) Za vsak  $i$ , kjer je  $s \leq i < k$ , v permutirani tabeli velja  $A[i] \leq p$ .
  - ii) Za vsak  $i$ , kjer je  $k < i \leq t$ , v permutirani tabeli velja  $A[i] \geq p$ .
  - iii) V permutirani tabeli velja  $A[k] = p$ .

Poleg tega mora algoritem uporabiti le konstantno količino dodatnega spomina (poleg spomina tabele  $A$ ) in mora imeti časovno zahtevnost  $O(n)$ .

- b) Kaj naredi rekurziven algoritem  $f(A,s,t,q)$  na Sliki 1, za  $q$  v intervalu  $1 \leq q \leq n$ ?
- c) Recimo, da so vse vrednosti v tabeli  $A$  različne. Dokažite, da je pričakovana časovna zahtevnost algoritma  $f(A,s,t,q)$  reda  $O(n)$ .
- d) Kakšna je pričakovana časovna zahtevost  $f(A,s,t,q)$ , če so vse vrednosti v tabeli  $A$  enake?

In the questions below  $A$  is an integer array that is defined for all indices  $i$  in the range  $s \leq i \leq t$ , and we write  $n$  for the size of the array; i.e.,  $n = t - s + 1$ .

- a) Write an algorithm  $\text{partition}(A,s,t,p)$ , where  $p$  is any integer such that  $p = A[i]$  for some index  $i$  with  $s \leq i \leq t$ . The algorithm is required to permute the values of the array  $A$  *in place* (i.e., the final permutation must reside in the same array  $A$ ) and return an integer  $k$  such that  $s \leq k \leq t$  and the following properties hold.
- i) For all  $i$  with  $s \leq i < k$ , we have  $A[i] \leq p$  in the permuted array  $A$ .
  - ii) For all  $i$  with  $k < i \leq t$ , we have  $A[i] \geq p$  in the permuted array.
  - iii)  $A[k] = p$  in the permuted array.

Furthermore, the algorithm must only use a constant amount of additional memory (beyond that used to store the array  $A$ ), and its time complexity must be  $O(n)$ .

- b) What task does the recursive algorithm  $f(A,s,t,q)$  defined in Slika 1 perform, for  $q$  in the range  $1 \leq q \leq n$ ?
- c) Suppose all entries of the matrix  $A$  are distinct. Prove that the expected time complexity of  $f(A,s,t,q)$  is  $O(n)$ .
- d) What is the expected time complexity of  $f(A,s,t,q)$  if all entries of the matrix  $A$  are equal?

## 10. naloga (6 točk/marks)

Naj bo  $p$  polinom stopnje  $\leq n$ , ki interpolira točke

$$(x_0, 0), (x_1, 0), \dots, (x_{n-1}, 0), (x_n, 1)$$

s paroma različnimi abscisami. Dokažite, da je

$$p^{(n)}(x_n) = \frac{n!}{\omega'(x_n)},$$

kjer je  $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$ .

Let  $p$  be a polynomial of degree  $\leq n$  that interpolates the points

$$(x_0, 0), (x_1, 0), \dots, (x_{n-1}, 0), (x_n, 1)$$

with pairwise distinct abscissae. Prove that

$$p^{(n)}(x_n) = \frac{n!}{\omega'(x_n)},$$

where  $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$ .

## 11. naloga (6 točk/marks)

Igralca A in B mečeta kocko. Igralec A postane srečen, ko prvič pade liho število pik, igralec B pa, ko prvič kocka pokaže šest pik. Naj bo  $X$  število potrebnih metov, da postane srečen vsaj eden od njiju in  $Y$ , da postaneta srečna oba. Izračunajte

- a)  $E(X),$
- b)  $E(Y),$
- c)  $\text{Cov}(X, Y).$

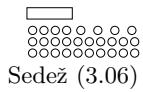
Players A and B throw a dice. Player A becomes happy when the die first shows an even number of spots. Player B becomes happy when the die first shows a six. Let  $X$  be the number of throws needed for at least one of the players to become happy, and let  $Y$  be the number of throws needed for both players to become happy. Calculate:

- a)  $E(X),$
- b)  $E(Y),$
- c)  $\text{Cov}(X, Y).$

**Preddoktorski izpit (Predoctoral exam)**  
25. aprila 2022 (25th April 2022)

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Ime in priimek



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Vpisna številka

Prostor za rešitve 5. do 11. naloge  
Space for solutions to questions 5 to 11





