Predoctoral exam

22nd of May 2015

There are 180 minutes of available time. Each problem is marked with 0, 1, 2 or 3 points. For a passing grade you have to collect at least 15 points. You must substantiate your answers with proofs or explanations, unless specifically stated otherwise. Good luck!

Problem 1 (3 points)

We drop a ball from a height of 1 m and let it bounce. What is the total distance traveled by the ball if on every bounce it reaches two thirds of the height of the previous bounce?

Problem 2 (3 points)

Is $\sqrt{3} - \sqrt{2}$ a rational number?

Problem 3 (3 points)

Let V be the vector space of real polynomials of degree ≤ 3. Give an example of a linear map of rank 2 from V to the vector space $\mathbb{R}^3$. What is the dimension of its kernel?

Problem 4 (3 points)

Prove that every real polynomial $p : \mathbb{R} \to \mathbb{R}$ whose degree is even has a *global* extremum.

Problem 5 (3 points)

Let $n \geq 2$ and let M_n be the vector space of real matrices of dimension $n \times n$. Does the subset $\{A \in M_n \mid A^2 = 0\}$ form a subspace?

Problem 6 (3 points)

Among all the differentiable functions $f : [0,1] \to \mathbb{R}$ satisfying $f(0) = 0$ and $f(1) = 1$ find the one for which the value of $\sup_{x \in (0,1)} |f'(x)|$ is minimal.

Problem 7 (3 points)

Solve *exactly one* of the following problems.

1. Let T be a random variable distributed exponentially with parameter λ. Without using integrals, express the conditional expectation $E(T \mid T \leq a)$ in terms of $E(T)$, where $a > 0$.
2. Give an example of a sequence of continuous maps $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, such that each one has maximal value 1, but pointwise they converge to the zero function.
3. We say that a table $[a_1, \ldots, a_n]$ is a *permutation*, if every number $1, \ldots, n$ appears in it exactly once. Write down *as efficient as possible* an algorithm or a program which determines whether a given table of integers is a permutation, and determine its time complexity with respect to the size of the table.

Problem 8 (3 points)

Solve *exactly one* of the following problems.

1. Let A and B be disjoint events for a given experiment. What is the probability that A happens before B in an infinite sequence of independent repetitions of the experiment?

2. Is the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f(x, y) = (x^2 + y + 1, x^4 + y^2 + x),$$

   a contraction with respect to the euclidean metric on $\mathbb{R}^2$? NB: a mapping f is a contraction if there exists a number $0 \leq \lambda < 1$, such that $\|f(u) - f(v)\| \leq \lambda \cdot \|u - v\|$ for all $u, v \in \mathbb{R}^2$.

3. We are given a function

```
def f(n, k):
  if n == 0: return 0
  else: return (f(n // k, k) + (n % k))
```

   where `a//b` is integer division and `a % b` remainder when `a` is divided by `b` (e.g., `13//5` is `2` and `13%5` is `3`).

   (a) What does `f(n,k)` calculate for integers `n` ≥ 0 and `k` ≥ 2?

   (b) How many additions happen when we call the function `f(n,k)`, as a function of `n` and `k`? (You may use the "big O" notation.)

Problem 9 (3 points)

Solve *exactly one* of the following problems.

1. Let N be a random variable distributed according to the Poisson distribution with parameter λ. If $N = n \in \mathbb{N}$ we throw a coin n times.

   (a) What is the distribution of the number of tails we observe?

   (b) Suppose we got k tails. What is the conditional distribution of the outcomes of N?

2. Prove that a group G is abelian if $(xy)^2 = x^2y^2$ for all $x, y \in G$. Then determine whether the following holds for all groups G and all $n \geq 2$: if $(xy)^n = x^ny^n$ for all $x, y \in G$ then G is abelian.

3. Initially we have 0 points and an unlimited supply of tiles labeled by number 2. We may join two tiles both labeled by a into a single tile labeled by $2a$, and we receive $2a$ points for doing so. For instance, by joining two tiles labeled by 16 we obtain a single tile labeled by 32 and receive 32 points. At least how many point have we got at the moment when we are in possession of a tile labeled by 2^k?

Problem 10 (3 points)

Solve *exactly one* of the following problems.

1. We consider an experiment with $n \geq 4$ outcomes which all equally likely. Let $K \subseteq S$ with at least $k \geq 3$ elements and $a \in K$. Let X_a be the random variable which counts the number of outcomes that are in K at the moment when a happens for the first time in a sequence of independent repetitions of the experiment. Calculate the probability $P(X_a = \ell)$ for $0 \leq \ell \leq k-1$.

2. Find all quadruples of numbers $a, b, c, d \in \mathbb{R}$ satisfying the following condition: for all infinitely differentiable maps $f : \mathbb{R} \to \mathbb{R}$ and all $x \in \mathbb{R}$ the limit

$$\lim_{h \to 0} \frac{f(x + a \cdot h) + b \cdot f(x + h) + c \cdot f(x)}{h^d}$$

   exists and its value equals a higher derivative $f^{(k)}(x)$ at x for some $k \geq 1$. NB: $f^{(k)}$ denotes the k-th derivative of f, where $f^{(0)} = f$.

3. Prove that for every natural number n there exists a simple graph on $4n$ vertices which is isomorphic to its complement.