Predoctoral exam

14th of December 2015

There are 150 minutes of available time. Each problem will be given a mark of 0, 1, 2 or 3 points. For a passing grade you have to collect at least 15 points. You must substantiate your answers with proofs or explanations, unless specifically stated otherwise. Good luck!

Problem 1 (3 points)

Is the series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1}$$

convergent? Explain your answer!

Problem 2 (3 points)

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function whose derivative is non-zero at every point of the interval. Does f necessarily have global extrema on this interval? In which points can f have its global extrema?

Problem 3 (3 points)

Let V_1 and V_2 be subspaces of a vector space V. Prove that their union $V_1 \cup V_2$ is a subspace only if $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Problem 4 (3 points)

Let f and g be functions from a set S to S. Suppose that $f \circ g = \mathrm{id}$. Prove that if S is a finite set, then $g \circ f = \mathrm{id}$, too. Show, by giving an example, that this does not hold for infinite sets.

Problem 5 (3 points)

Let $f(x) = 1 - x - \cos x + \sin x$. Prove:

1. the function f has at least one zero on the interval $(0, 2\pi)$;
2. the function f has exactly one zero on the interval $(0, 2\pi)$.

Problem 6 (3 points)

Let A be a complex $n \times n$ matrix. Show that $A^{n+1} = 0$ implies $A^n = 0$.

Problem 7 (3 points)

Solve *exactly one* of the following problems.

1. Is the polynomial $X^3 - 3$ irreducible as an element of the ring $\mathbb{Q}[X]$? What about as an element of $\mathbb{R}[X]$?

2. We want to tile a rectangular floor of dimension $2 \times n$ with tiles of two types: squares of dimension 2×2 and rectangles of dimension 1×2 (tiles can be rotated arbitrarily). In how many ways can this be done? If you cannot provide a general answer, answer it for $n = 12$.

3. A student has to pass a multiple choice test consisting of ten questions where four answers are offered to each question. To five questions he knows the answer, to two questions he can identify two wrong answers and for the other three he has no clue. What is the expected number of correct answers of the student? What is the probability that he answers exactly 9 questions correctly?

Problem 8 (3 points)

Solve *exactly one* of the following problems.

1. Let $f_n\colon [0,1] \to [1,2]$ be a sequence of linear functions. Prove that there exists a subsequence that converges uniformly on the interval $[0,1]$.

2. Consider the two recursive programs below, which both define the same mathematical function.

   ```
   foo1(x) =  if x==1 then 1 else foo1(x div 2) + foo1(x div 2)
   foo2(x) =  if x==1 then 1 else 2 * foo2(x div 2)
   ```

   Here `==` tests equality, and `div` is the integer division function $m, n \mapsto \lfloor m/n \rfloor$, which returns the integer part of a division, and discards the fractional part.

   (a) Write a non-recursive formula for the mathematical function on positive natural numbers $\mathbb{N}^+$ computed by both the `foo` functions.

   (b) Assuming function calls are implemented naively (no memoization), write non-recursive formulas for functions $t_1, t_2 : \mathbb{N}^+ \to \mathbb{N}^+$ such that $t_1(n)$ counts the number of function calls made in executing `foo1`(n), and $t_2(n)$ counts the number of function calls made in executing `foo2`(n).

3. Two dice are rolled. The event A is "the first dice is even", the event B "the second dice is even" and event C "the sum of the dice is even". Are these events independent?

Problem 9 (3 points)

Solve *exactly one* of the following problems.

1. Prove that
$$d(A,B) = \operatorname{rank}(A - B)$$
defines a metric on the set of all $m \times n$ matrices.

2. For positive integers n and k, let $c(n,k)$ denote the number of permutations of the set $\{1, 2, \ldots, n\}$ which can be written as a product of precisely k disjoint cycles (counting cycles of length 1 too). Find explicit formulae for $c(n,1)$ and $\sum_{k=1}^{n} c(n,k)$, and find a recursive formula for $c(n,k)$ expressing $c(n,k)$ as a function of n, k, $c(n-1,k-1)$ and $c(n-1,k)$.

3. Let $T \overset{(d)}{=} Exp(\lambda)$ and $S \overset{(d)}{=} Exp(\mu)$ be two independent random variables. Compute the density of $T + S$ in cases:
   (a) $\lambda \neq \mu$;
   (b) $\lambda = \mu$.

Problem 10 (3 points)

Solve *exactly one* of the following problems.

1. Endow the vector space P of all polynomials with real coefficients by the norm $\|p\| = \max_{0 \le x \le 1} |p(x)|$. Is the map $D : P \to P$, $D(p) = p'$ continuous with respect to this norm?

2. A *nondeterministic* Turing machine N is said to *accept* its input if it has at least one possible execution sequence that terminates in an accepting state. Is it possible to construct a *deterministic* Turing machine M that accepts the same set of inputs as N? Justify your answer.

3. A random variable has density $\frac{x}{2}\mathbf{1}_{[0,2]}(x)$.
   (a) What is the probability that it differs from its mean by less than its standard deviation.
   (b) The same question, if mean is replaced by median.